CSCI 7000-019 Fall 2023: Problem Set 6
Counting with Complexity
Due: Monday Nov 6, 2023

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A decision problem $A \colon \{0,1\}^* \to \{0,1\}$ is *(polynomially) sparse* if there is some polynomial p such that, for all n,

$$|\{x \in \{0,1\}^{\leq n} : A(x) = 1\}| \leq p(n).$$

1. (a) If A, B, C are decision problems, and $A \leq_m^p B$ and $B \leq_m^p C$, show that $A \leq_m^p C$.

   (b) A decision problem A is NP-hard if every $B \in \mathsf{NP}$ reduces to A ($B \leq_m^p A$). A decision problem is NP-complete if it is NP-hard and also in NP. Boolean Satisfiability is NP-complete, as is Graph 3-Colorability. Show that if A is NP-hard and $\mathsf{P} \neq \mathsf{NP}$, then A is not sparse.

   (c) If C is NP-complete, show that you can use Mahaney's Theorem with C in place of SAT. *Hint:* Do *not* re-prove Mahaney's Theorem, just combine the right ingredients to show that it applies directly as a black box.

Consider the following decision problems:

3-SAT
Input: A Boolean 3-cnf φ, that is, $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where each C_i is an OR of three literals (a literal can be either a variable or a negated variable)
Decide: Is φ satisfiable?

3COL
Input: A simple undirected graph G.

Decide: Is G properly 3-colorable? That is, is there an assignment of colors $c\colon V(G) \to \{r, g, b\}$ such that for all $(u, v) \in E(G)$, $c(u) \neq c(v)$?

HamiltonianCycle
Input: A directed graph G
Decide: Does G contains a Hamiltonian cycle? That is, is there a directed cycle in G such that every vertex of G occurs exactly once in the cycle?

2. You may take for granted that 3SAT is NP-complete.
   (a) Show that 3COL is NP-complete.
   (b) Show that, if $\mathsf{P} \neq \mathsf{NP}$, there are super-polynomially many graphs on n vertices that are properly 3-colorable, and super-polynomially many that are not properly 3-colorable, using Mahaney's and Fortune's Theorems.
   (c) Same as (b), but by directly constructing that many instances of 3-colorable and non-3-colorable graphs.

3. You may take for granted that 3SAT is NP-complete.
   (a) Show that Directed Hamiltonian Path is NP-complete.
   (b) Show that, if $\mathsf{P} \neq \mathsf{NP}$, there are super-polynomially many directed graphs on n vertices that have Hamiltonian cycles, and super-polynomially many that don't, using Mahaney's and Fortune's Theorems.
   (c) Same as (b), but by directly constructing that many graphs.